Monte Carlo tree search (MCTS) methods have had recent success in games, planning, and optimization. MCTS uses results from rollouts to guide search; a rollout is a path that descends the tree with a randomized decision at each ply until reaching a leaf. MCTS results can be strongly influenced by the choice of appropriate policy to bias the rollouts. Most previous work on MCTS uses static uniform random or domain-specific policies. We describe a new MCTS method that dynamically adapts the rollout policy during search, in deterministic optimization problems. Our starting point is Cazenave’s original Nested Monte Carlo Search (NMCS), but rather than navigating the tree directly we instead use gradient ascent on the rollout policy at each level of the nested search. We benchmark this new Nested Rollout Policy Adaptation (NRPA) algorithm and examine its behavior. Our test problems are instances of Crossword Puzzle Construction and Morpion Solitaire. Over moderate time scales NRPA can substantially improve search efficiency compared to NMCS, and over longer time scales NRPA improves upon all previous published solutions for the test problems. Results include a new Morpion Solitaire solution that improves upon the previous human-generated record that had stood for over 30 years.

1 Introduction
Monte Carlo tree search (MCTS) methods have had substantial recent success in two-player games [Gelly et al., 2007; Finnsson et al., 2010], planning [Nakhost et al., 2009; Silver et al., 2010], optimization and one-player games [Cazenave, 2009; Rimmel et al., 2011; Méhat et al., 2010] and practical applications [de Mesmay et al., 2009; Cazenave et al., 2009]. In this paper, we focus on deterministic optimization problems. For these, nested Monte Carlo search has been particularly successful, with world-record results in several problems [Cazenave, 2009; Bjarnason et al., 2007].

MCTS guides search using results from rollouts. The base rollout policy is a key factor in MCTS success. Most prior MCTS work uses static policies, but some work has appeared on adapting rollout policies in two-player games [Silver et al., 2009; Tesauro et al., 1996; Finnsson et al., 2010]. One recent study uses an evolutionary algorithm to choose policy parameters [Rimmel et al., 2011]. Methods for adapting rollout policies also exist in control and reinforcement learning [Bertsekas, 1997; Fern et al., 2003; Veness et al., 2011].

In this paper, we describe a new MCTS algorithm that adapts its rollout policy during search. It resembles Cazenave’s Nested Monte Carlo Search (NMCS) [Cazenave, 2009], in that every search step advances towards the current best known solution. But whereas NMCS makes these moves directly on the search tree, here we instead adapt a rollout policy towards the current best solution.

Section 2 describes the new algorithm. Section 3 describes the test problems: two Crossword Puzzle Construction instances for which the search tree has large branching factor but relatively small depth, and two versions of Morpion Solitaire with lower branching factor but larger depth. Section 4 has experimental results and comparison to NMCS.

2 Algorithm
We start with Cazenave’s Nested Monte Carlo Search (NMCS) [Cazenave, 2009], and then proceed to the new Nested Rollout Policy Adaptation (NRPA) algorithm. For both, search is initiated at a nesting level $n$, which is the depth of recursion for the nested search. At each nesting level, the algorithm searches for a solution sequence (a sequence of child selections defining a path descending the tree to a leaf). In searching for improvements to its current best sequence, nesting level $n$ makes recursive calls to nesting level $n - 1$, bottoming out in the base rollout policy at level 0. When nesting succeeds, level $n$ returns substantially higher-quality solutions than level $n - 1$ [Cazenave, 2009].

2.1 Nested Monte Carlo Search
Fig. 1 shows NMCS. Lines 2-9 are the level 0 rollout from a given node of the search tree; a uniform random policy is shown. Nesting levels $n \geq 1$ start from a given node and iterate through successive ply (lines 14 and 22-23). Each iteration recursively calls level $n - 1$ on each child of the current node (lines 15-17), replacing the current best solution from the current ply to the end if an improvement is found (lines 18-21). Each iteration concludes by advancing the current node one ply towards the current best solution (line 22).
Figure 1: NMCS with Uniform Rollout Policy

An NMCS search tree is defined by operations root(), num_children(node), child(node,i) returning the i'th child of node, and score(node) at a leaf. Search at nesting level L is initiated by NMCS(L.root()). The only tunable parameter, nesting level, has just a few feasible settings. More domain-specific tuning can be done by reshaping the tree [Bjarnason et al., 2007] and choosing a heuristic level 0 policy [Bjarnason et al., 2007; Cazenave, 2007; Rimmel et al., 2011].

2.2 Nested Rollout Policy Adaptation

Fig. 2 shows the new Nested Rollout Policy Adaptation (NRPA) algorithm. Lines 2-9 are the level 0 rollout, which follows (lines 5-6) a given weighted policy from the root to a leaf. Nesting levels \( n \geq 1 \) do a fixed number of iterations (line 13) starting from an initial given policy. Each iteration uses the current policy as a starting point in a recursive call to level \( n-1 \) (line 14), replacing the current best solution when a tie or improvement is found (lines 15-17), then advancing the policy one gradient ascent step towards the current best solution sequence (line 18 and Adapt()); see Section 2.3). Note level \( n-1 \)’s final policy is not returned to level \( n \).

Search at nesting level \( L \) is initiated by NRPA(L,pol) with all-zero vector pol. An NRPA search tree is defined by the same operations as NMCS, with the addition of code(node,i) which returns a domain-specific code for the action leading from the node to its i’th child. This provides the domain-specific representation for policy adaptation. No further domain-specific tuning of the policy is available; it is determined by adaptation starting from a uniform random policy. In addition to the nesting level parameter, NRPA allows tuning of Alpha and number of iterations \( N \) per level. Results here use Alpha=1.0 and \( N=100 \), across the test problems.

2.3 NRPA's Design

NMCS can be applied to any search tree, even when nodes and children have no known properties beyond their place in the tree. But many problems have additional structure. NRPA benefits when actions with the same code have similar value across a range of nodes. The general idea, that actions can have similar value in a range of contexts, has helped improve

Figure 2: NRPA with Global Parameters Alpha and N
solution sequence so far until it is overwhelmingly likely to recapitulate that solution sequence. Adapt() can increase this probability exponentially quickly: very roughly, one call increases probability by a factor of \( e^{\alpha \text{t}} \) on each action that still has low probability. This rapid increase in sequence probability has modest dependence on branching factor and sequence length, and in Section 4 a single value of \( \alpha \) works well across a range of problems.

This convergence towards recapitulation depends upon NRPA adapting only towards a single best-so-far solution. If it was based instead on a mix of solutions, representational limitations might cause convergence towards a suboptimal compromise amongst multiple solutions, preventing recapitulation of any one best-so-far solution score. Nonetheless, we do replace the current best solution with a new equally good one that ties its score. This preserves the property of return to the best score found so far, while aiding exploration. In preliminary experiments, taking ties gave NRPA substantially better results, compared to only taking strict improvements.

NRPA can be compared to other distribution-based search methods [Larranaga et al., 2002].

### 2.4 Choosing Policy Representations for NRPA

The following two properties suffice so that NRPA’s policy can converge towards any sequence. First, for any node \( s \), code\((s, i)\) should be distinct so that any child can be assigned arbitrarily high probability. Second, any solution sequence with nodes \( s_t \) in which children \( i_t \) are selected should have code\((s_t, i_t)\) distinct from code\((s_{t'}, i_t)\) for all \( t' > t \) and all \( i \). That is, selecting \( i_t \) in \( s_t \) “retires” code\((s_t, i_t)\), so putting high probability on \( i_t \) doesn’t interfere with later action selection.

In addition, if any legal permutation of a sequence code\((s_t, i_t)\) yields the same score, NRPA can explore permutations more freely with less need to constrain ordering.

The test problems here each have a simple and natural representation that adheres to the above properties. Our final results use the same representations that were initially tested; no exploration of representational alternatives was needed.

Other problems may benefit from exploration of representational alternatives, as is often the case in machine learning. For NRPA, actions with same code should have similar value. But this only needs to be approximately true, and only within limited scope (as opposed to the entire state space). Actions with distinct codes are searched separately, so the total number of available codes should be minimized. Some problems may benefit from code\((s, i)\) including limited additional context from global state \( s \) [Finnsson et al., 2010].

### 3 Test Problems

#### 3.1 Crossword Puzzle Construction

An instance of Crossword Puzzle Construction (CPC) provides a grid and a dictionary of allowed words. A feasible solution is a legal crossword. A legal crossword places a subset \( S \) of the dictionary on the grid, with words oriented either horizontally left-to-right or vertically top-to-bottom, such that every two adjacent occupied grid positions are part of one of the placed words from \( S \).

### Table 1: Test Problems

<table>
<thead>
<tr>
<th></th>
<th>Depth</th>
<th>Branch</th>
<th>Old Record</th>
<th>New</th>
</tr>
</thead>
<tbody>
<tr>
<td>MorpT</td>
<td>177</td>
<td>46</td>
<td>170 [Bruneau, 1976]</td>
<td>177</td>
</tr>
<tr>
<td>MorpD</td>
<td>82</td>
<td>46</td>
<td>80 [Cazenave, 2009]</td>
<td>82</td>
</tr>
<tr>
<td>CrossC</td>
<td>37</td>
<td>1197</td>
<td>36;256 [CrossC, 2006]</td>
<td>37;259</td>
</tr>
<tr>
<td>CrossP</td>
<td>28</td>
<td>988</td>
<td>28;236 [CrossP, 1994]</td>
<td>28;238</td>
</tr>
</tbody>
</table>

Instances here have a square grid with all positions open to use. Heuristic search has previously been applied [Ginsberg et al., 1990] to a generalized decision version of CPC which is NP-complete [Garey and Johnson, 1979].

Instances here are from *GAMES Magazine* contests. Contests were diverse (not all CPC). Contests were approachable by humans without computer assistance, but computer assistance was allowed. Contests included monetary prizes, and attracted high-quality solutions. From over 12 years of contests, we select two CPC instances with largest grid (19x19).

**MorpC** has 63 allowed words: 50 state, 10 provincial, and 3 territorial capitals in the U.S. and Canada with spaces, punctuation, and accents removed (e.g. Quebec City becomes QUEBECITY) [CrossC, 2006]. **CrossP** has 52 words: the 52 U.S. National Parks listed in *The 1994 World Almanac* with spaces and punctuation removed [CrossP, 1994]. Both yielded a unique winner (Table 1) from hundreds of entries.

Nodes in the CPC search tree are feasible solutions as defined above, starting with zero words at the root and adding one word per ply until reaching leaves in which no additional words can be added. The first ply adds an initial word, constrained to be vertical and start in the top row. This is without loss of generality: any solution that uses the top row must have a vertical word starting there, and any solution that does not use the top row can be shifted upwards without changing score. Beyond the first word, children of a node consist of all single words which may be added to leave a new legal state.

For NRPA, code\((node, i)\) identifies the word added by child \( i \) together with its grid position and orientation.

#### 3.2 Morpion Solitaire

Morpion Solitaire [Boyer, 2011] is a pencil-and-paper puzzle on a square grid that is presumed to be infinite. Given a current configuration of dots (marked intersections on the grid), a legal turn requires the addition of one more dot such that...
Morpion Solitaire is NP-hard [Demaine et al., 2006]. This paper is the first to describe NRPA, and the record of 177 was set by NRPA in 2010 [Boyer, 2010]. But a 1976 human-generated MorpT record of 170 [Bruneau, 1976] stood for over 30 years, until NRPA obtained 172 in August 2010 [Boyer, 2010; 2011]. This paper is the first to describe NRPA, and the record is improved to 177 in this paper.

Our search tree has the starting configuration at the root. Children of a node consist of all legal distinct lines that may be added to the grid; such a line uniquely determines the dot that is added. Leaves of the tree have no further legal moves.

We modify the tree slightly by replacing a type of dominated move. In Fig. 4, two moves add a new dot between 0 and 1: move A extends from 0 to 3, and move B from the new dot to 4. Move A blocks line segments 3-4 and 4-5; after move A these cannot be used by later moves. Move B blocks only 4-5. Move B occupies 3-4 whereas A does not, but since A blocks 3-4 it does not gain any advantage by leaving 3-4 unoccupied. Move A is dominated by B: any legal sequence of moves that includes A is still legal if A is replaced by B, and B may open up more possibilities (by leaving the segment from 0 to the new dot available). During NMCS and NRPA, nodes with an available dominated move (like A) retain the same number of children, and the dominated move may be selected and recorded as part of the solution sequence, but its effect is replaced by that of the dominating move (like B). This gave a small improvement in early experiments, more so than removing dominated moves entirely.

For NRPA, code(node,i) uniquely identifies the endpoints of the chosen line to be added by child i. In the case of a dominated move, it still codes the line represented by the dominated move (even though the line actually placed is different). This ensures all children have a unique code even when dominated moves exist. We use a 64x64 grid with the initial configuration near the center, which sufficed so that search did not approach the edges of the grid.

### 4 Experimental Results

Table 1 summarizes the test problems. “Depth” and “Branch” are maximum depth and branching factors seen in a set of test runs. For CPC, maximum branching occurs at the root where any word can be placed at the top of any column, but large branching factors of over 400 occur deeper in the tree as well.

#### 4.1 Comparing Efficiency of Search

We wish to compare the effectiveness of NMCS and NRPA. To account for the methods’ differing computation time and overhead, we compare based on actual execution times measured on a reference machine (3.6GHz Core i7 with 4 physical cores; at most 4 single-threaded jobs run at any time).

A fixed time horizon is used (1000 or 10000 seconds). If a search configuration completes before the time horizon is exhausted, the time horizon is filled with independent restarts of the same configuration. At each point in time, we track the max score seen so far among all restarts of the search configuration. This yields a single timeline for the search configuration; at least 15 independent timelines are generated for each configuration. Multiple independent timelines can then be combined to form a picture of the typical trajectory. This has been used to illustrate that NMCS typically becomes more efficient as nesting level increases [Cazenave, 2009], and NRPA shows a similar trend (Fig. 5). When displaying results, we combine multiple independent timelines at each time t by taking the median (defined here as the mid element in a sorted list, or the larger of the 2 mid elements of an even-length list). When comparing samples, we use a Mann-Whitney U test of statistical significance with $p < 0.05$.

Given a time horizon, we ran NRPA and NMCS at increasing nesting level, until it was no longer possible to complete the majority of a single search run within the horizon. NMCS consistently gave best results at the largest nesting level, and so this is the result we show. In some cases, NRPA can run at a larger nesting level than NMCS in the same horizon. Then, we show NRPA results at both the larger level as well as the same nesting level at which NMCS ran.

Table 2 shows median results at 100 and 1000 seconds for NRPA and NMCS. For all test problems, NRPA results at
1. If the previous move resulted in new lines becoming available: sample 4 such newly-available lines (with replacement) and pick one yielding the largest number of total available moves [Cazenave, 2007].

2. Otherwise, with 90% probability: from available moves which place their new dot at horizontal/vertical/diagonal distance 1 or 2 from the previously-added dot, sample 2 (with replacement) and pick the one yielding more total available moves.

3. Otherwise, sample 6 available moves (without replacement; take all if fewer than 6 exist) and pick one yielding the largest number of newly-available lines as described in 1 above, breaking ties via the largest number of total available moves.

Figure 6: Tuned MorpT Rollout Policy

100 seconds are statistically significantly better than NMCS at 1000 seconds. So, in these tests, NRPA consistently obtains better results than NMCS, even in 1/10 the time used by NMCS. This was true whether NRPA was run at its highest nesting level, or at the same nesting level as NMCS.

**Sample-k NMCS**

For CPC, NMCS spends a large fraction of its time iterating through high branching factors near the search tree’s root. This may not be the best use of limited time, and it reduces the nesting level that can fit in the time horizon. So, we also test an alternative “Sample-k NMCS” which at each node samples (with replacement) a fixed number k of children, regardless of branching factor. With k sufficiently low, Sample-k NMCS can fit higher nesting levels than NMCS in the same time horizon. Table 3 shows median results for CrossC with a time horizon of 10000 seconds that allows Sample-5 NMCS to run at nesting level 4 and Sample-15 to run at nesting level 3. Sampling enables improved results compared to standard NMCS at nesting level 2. But NRPA level 3 at 1000 seconds still gives statistically significantly better results than Sample-k NMCS (and standard NMCS) at 10000 seconds. Again, NRPA obtains better results in 1/10 of the time.

Table 3: CrossC Median Scores with Sampled NMCS

<table>
<thead>
<tr>
<th>Method</th>
<th>Level</th>
<th>$10^2$ sec</th>
<th>$10^3$ sec</th>
<th>$10^4$ sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>NRPA</td>
<td>3</td>
<td>33;229</td>
<td>34;244</td>
<td>35;247</td>
</tr>
<tr>
<td>NMCS</td>
<td>2</td>
<td>31;223</td>
<td>32;229</td>
<td>33;235</td>
</tr>
<tr>
<td>Sample-5 NMCS</td>
<td>4</td>
<td>32;228</td>
<td>33;232</td>
<td>34;235</td>
</tr>
<tr>
<td>Sample-15 NMCS</td>
<td>3</td>
<td>32;231</td>
<td>33;235</td>
<td>33;248</td>
</tr>
</tbody>
</table>

**Tuned NMCS Rollout Policy**

Since NRPA adapts its rollout policy, it is natural to compare NMCS using a domain-specific policy. Before running NRPA on MorpT, we manually tuned a rollout policy via small experiments evaluating policy elements’ impact on results.

Table 4: MorpT Median Scores with Tuned NMCS

<table>
<thead>
<tr>
<th>Method</th>
<th>Level</th>
<th>$10^2$ sec</th>
<th>$10^3$ sec</th>
<th>$10^4$ sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>NRPA</td>
<td>4</td>
<td>152</td>
<td>155</td>
<td>160</td>
</tr>
<tr>
<td>NRPA</td>
<td>3</td>
<td>153</td>
<td>157</td>
<td>158</td>
</tr>
<tr>
<td>NMCS</td>
<td>3</td>
<td>116</td>
<td>130</td>
<td>137</td>
</tr>
<tr>
<td>Tuned NMCS</td>
<td>3</td>
<td>137</td>
<td>146</td>
<td>149</td>
</tr>
</tbody>
</table>

The tuned policy (Fig. 6) is effective in NMCS; Table 4 shows at 10000 seconds it already obtains a median score higher than the best prior MorpT computer search result [Akiyama et al., 2010]. But, NMCS with the tuned policy is still much less efficient than NRPA, even though NRPA adapts from scratch and does not benefit from such manual tuning. NMCS, with or without the tuned policy, gives statistically significantly worse results at 10000 seconds than NRPA at 100 seconds (whether NRPA is run at level 3 or 4). So, NRPA obtains better MorpT results than NMCS in 1/100 of the time, even when NMCS is using a well-tuned policy.

**4.2 Results of Longer Runs**

We ran NRPA over longer times, to compare NRPA’s best solutions to the best previously published solutions for the test problems. At the nesting levels selected, approximate per-run reference machine time is 1 hour for MorpD, 15 hours for CrossP, 24 hours for CrossC, and 1 week for MorpT. As a rough comparison, best previous results were reported as taking 38 days for a MorpT run [Akiyama et al., 2010] and 10 days for a MorpD run on a single machine [Cazenave, 2009].

Table 5: NRPA Results of Longer Runs

<table>
<thead>
<tr>
<th>Method</th>
<th>Level</th>
<th>Runs</th>
<th>Median</th>
<th>Max(Rep)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MorpT</td>
<td>5</td>
<td>28</td>
<td>171</td>
<td>177 (2)</td>
</tr>
<tr>
<td>MorpD</td>
<td>4</td>
<td>40</td>
<td>82</td>
<td>82 (25)</td>
</tr>
<tr>
<td>CrossC</td>
<td>4</td>
<td>40</td>
<td>35;249</td>
<td>37;259 (1)</td>
</tr>
<tr>
<td>CrossP</td>
<td>4</td>
<td>40</td>
<td>27;234</td>
<td>28;238 (3)</td>
</tr>
</tbody>
</table>

Table 5 has results. For MorpT and MorpD, even the median is an improvement over the best previously published solutions. For CrossC and CrossP, the maximum is an improvement over the best previous (contest-winning) solutions (see Table 1). Fig. 7 has the new CrossC and MorpT records.

**4.3 Distribution of NRPA Results**

Fig. 8 has a distribution of results from NRPA nesting levels 0-5, for MorpT. Nesting level n required roughly 100x the time of $n - 1$, so on the logarithmic vertical axis we can roughly conclude nesting level n is more efficient than $n - 1$ if its cumulative histogram is over two decades higher. Nesting levels 1-3 show especially large efficiency gains.
4.4 Permutations and Hybrids

We examine the composition of intermediate solution sequences obtained by NRPA. From over 200 MorpT runs with nesting level 3, we gather each tie and improvement obtained at level 3 during its 100 iterations. For each, we examine the codes returned by code() for its sequence of actions, and categorize each code as one of: Prefix for the initial segment that exactly matches the previous solution sequence’s initial segment (note NMCS strictly increases this through a level’s iterations, but NRPA need not); Permutation matches the previous solution, but in a permuted order rather than as part of Prefix; Hybrid does not match the immediately previous solution, but does match an older solution at this level; and New were not used in any previous solution at this level.

Table 6: NRPA Intermediate Solution Content for MorpT

<table>
<thead>
<tr>
<th>Ties</th>
<th>Prefix</th>
<th>Permutation</th>
<th>Hybrid</th>
<th>New</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1%</td>
<td>89.7%</td>
<td>8.3%</td>
<td>1.9%</td>
<td>20.0%</td>
</tr>
</tbody>
</table>

Table 6 has results. NRPA freely explores permutations, with little of the initial segment staying the same. Solutions also have substantial Hybrid content; NRPA’s policy retains some emphasis on codes from older solutions (not just the current best). Hybrid content is higher in improvements than ties, suggesting it can contribute usefully to progress.

5 Conclusion

We have presented NRPA, an MCTS algorithm that uses gradient ascent on its rollout policy to navigate search. NRPA yields substantial search efficiency improvements as well as new record solutions on our test problems. NRPA is the first computer search method to improve upon a human-generated Morpion Solitaire record that had stood for over 30 years.

Ongoing work includes more complex applications, enabling code() to return a feature vector, and parallelization.

Acknowledgments

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References